Gröbner Bases of Structured Systems and Applications to Cryptology

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INRIA/CNRS/Univ. Lorraine, Caramel Project

Journées C2, March 26, 2014

Algebraic cryptanalysis

General framework

- Modeling of a **cryptosystem** by an algebraic **polynomial system**;
- Coefficients in a **finite field**;
- **Solving** $\longrightarrow$ retrieving secret information;
- **Complexity** $\longrightarrow$ security;
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Question: impact of **structures** on **GB computations**.

0-dimensional solving strategy with Gröbner bases

$f_1 = \cdots = f_m = 0$		**Complexity**	**Algorithms**
$\Downarrow$ "grevlex" Gb	**Row Echelon** forms of Macaulay matrices up to degree $\mathsf{d}_{\mathsf{reg}}$	$O\left(m\binom{n+\mathsf{d}_{\mathsf{reg}}}{n}^{\omega}\right)$	**Buchberger (1965)** F_4 **(Faugère 1999)** F_5 **(Faugère 2002)**
$\Downarrow$ "lex" Gb	**Linear algebra** in $\frac{\mathbb{K}[\mathbf{X}]}{I}$ as a $\mathbb{K}$-vect. space of dim. $\mathrm{DEG}(I)$ $\rightsquigarrow g(u) = 0, x_i = h_i(u)$	$O\left(n\mathrm{DEG}(I)^3\right)$	**FGLM** Faugère, Gianni, Lazard, Mora (1993)

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Macaulay matrix in degree d

$f_1 = \cdots = f_m = 0, \deg(f_i) = d_i$
Rows: all products tf_i where $t \in \mathrm{Monomials}(d - d_i)$.
Columns: monomials of degree d.

$$\begin{array}{c} \\ t_1 f_1 \\ \vdots \\ t_k f_m \end{array} \begin{array}{c} m_1 \succ \cdots \succ m_\ell \\ \left(\begin{array}{ccc} & & \\ & & \\ & & \end{array}\right) \end{array}$$

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Degree of regularity:
maximal degree reached

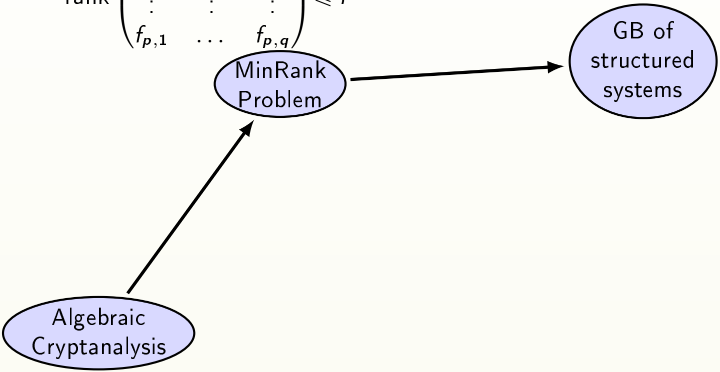
Hilbert series:
generating series of the rank defects

$$\mathrm{HS}(t) = \sum_{d \in \mathbb{N}} \dim(\mathbb{K}[X]_d / I_d) t^d$$

$$d_{reg} = \deg(\mathrm{HS}) + 1$$

Structured Systems in Cryptology

$$\operatorname{rank}\begin{pmatrix} f_{1,1} & \dots & f_{1,q} \\ \vdots & \vdots & \vdots \\ f_{p,1} & \dots & f_{p,q} \end{pmatrix} \leqslant r$$



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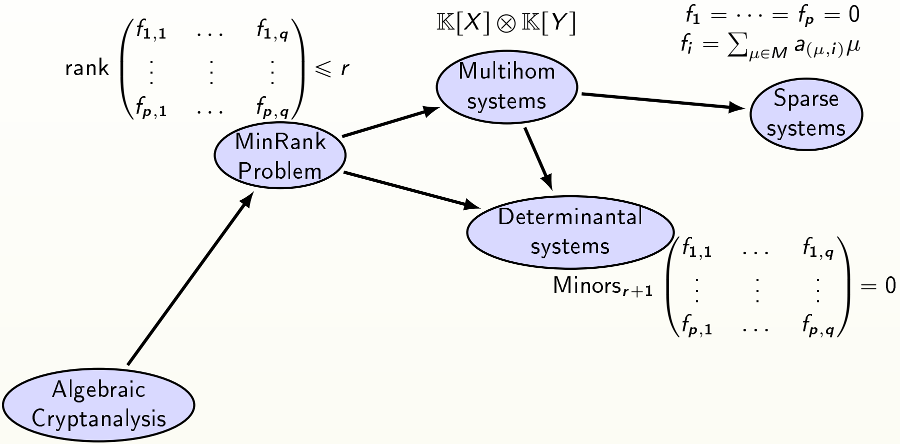

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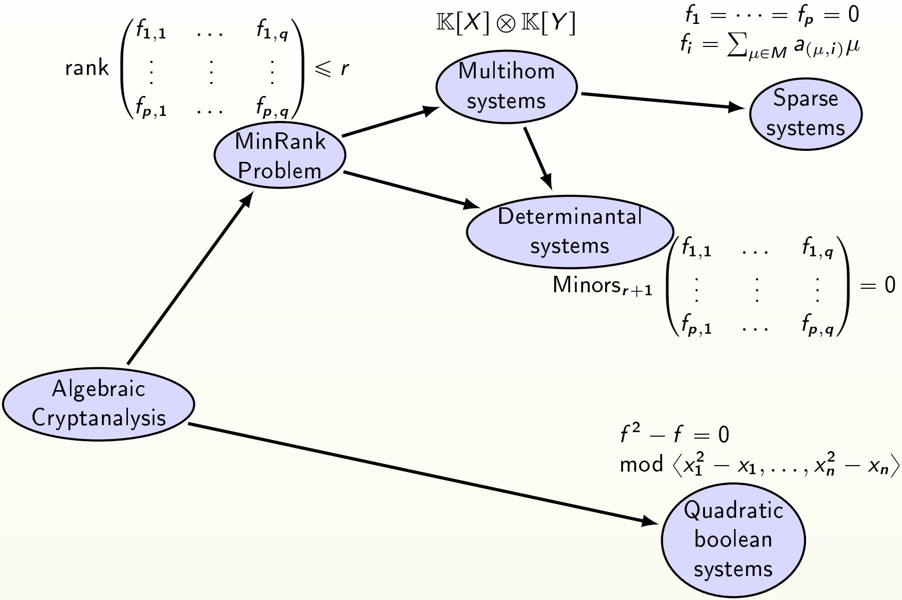

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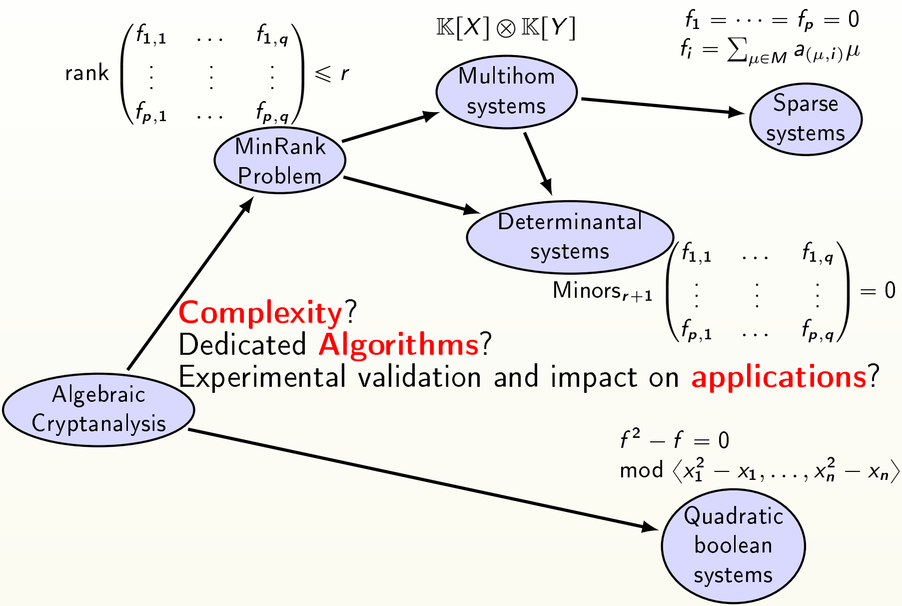

Complexity?
Dedicated **Algorithms**?
Experimental validation and impact on **applications**?

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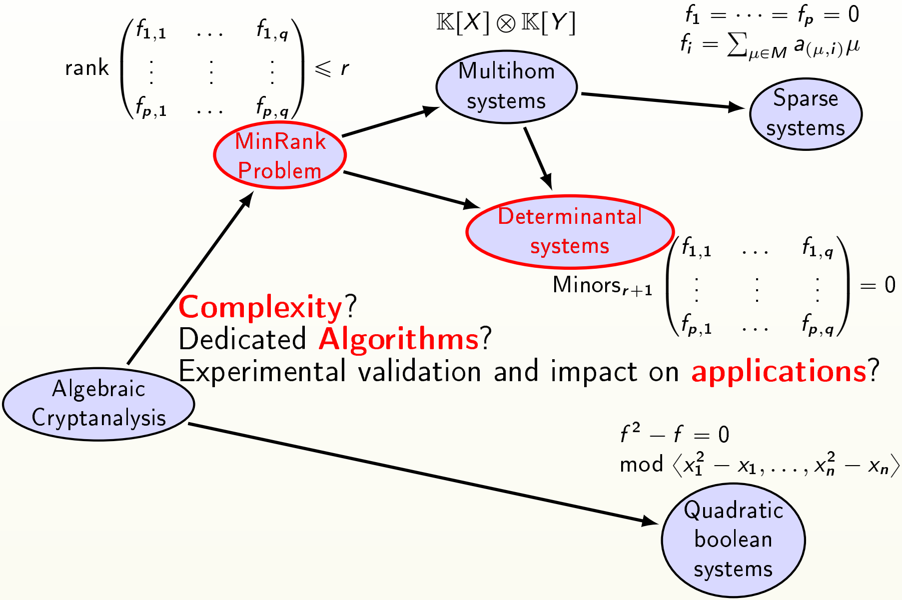

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MinRank

$r \in \mathbb{N}$. $M_0, \dots, M_n$: $n+1$ matrices of size $p \times q$.

MinRank Problem

Find $\lambda_1, \dots, \lambda_n$ such that

$$\mathbf{Rank}\left(M_0 - \sum_{i=1}^{n} \lambda_i M_i\right) \leqslant r$$

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- **Multivariate** generalization of the **EigenValue** problem.
- Applications in **cryptology**, **coding theory**, **geometry**, ...
  Kipnis/Shamir Crypto'99 Faugère/Levy-dit-Vehel/Perret, Crypto '08
  Courtois Crypto'01 Gaborit/Ruatta/Schrek'13
- Fundamental **NP-hard** problem of **linear algebra**.
  Buss/Frandsen/Shallit, 1999.

Determinantal systems

Let $r < q < p$ be integers and M be the $p \times q$ matrix

$$\mathrm{M}(X) = \begin{bmatrix} f_{1,1}(X) & \cdots & \cdots & f_{1,q}(X) \\ \vdots & \dots & \dots & \vdots \\ f_{p,1}(X) & \cdots & \cdots & f_{p,q}(X) \end{bmatrix}$$

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Generalized MinRank Problem

Compute the set of points $\mathbf{x} \in \overline{\mathbb{K}}^n$ such that $\mathrm{rank}(\mathrm{M}(\mathbf{x})) \leqslant r$.

$\rightsquigarrow$ **polynomial system solving** problem: $\mathrm{Minors}_{r+1}(\mathrm{M}(X)) = 0$

Main results

with J.-C. Faugère, M. Safey El Din

$p \times q$ matrix. n variables. Entries of degree D.
Zero-dimensional case ($n = (p-r)(q-r)$).

	System $\longrightarrow$ **grevlex GB**	$\underset{\text{Change of ordering}}{\longrightarrow}$ **lex GB**.
Complexity	$O\left(\left(\binom{\mathbf{p}}{\mathbf{r}+\mathbf{1}}\binom{\mathbf{q}}{\mathbf{r}+\mathbf{1}}\binom{\mathbf{n}+\mathbf{d}_{\mathbf{reg}}}{\mathbf{d}_{\mathbf{reg}}}\right)^{\omega}\right)$	$O\left(\mathbf{n}\cdot \mathrm{DEG}^3\right)$

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Degree and regularity (under genericity assumptions on the coefficients)

$$d_{\mathrm{reg}} = Dr(q-r) + (D-1)(p-r)(q-r) + 1$$

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$\rightsquigarrow$ families of Generalized MinRank Problems that can be solved in complexity **polynomial** in the **number of solutions**.

Application in Cryptology

Courtois, Crypto'01
Authentication scheme based on the difficulty of MinRank. Proposed parameters:
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q	security	FGb F_5+FGLM
6	2^{106}	2.8s
7	2^{122}	130s
11	2^{138}	238 days (est.) on 64 quadcore proc.

Bottleneck for $q = 11$: computing the input system.

Roadmap of the proof

$$\mathscr{D} = \text{Minors}_{r+1} \begin{pmatrix} v_{1,1} & \cdots & v_{1,q} \\ \vdots & \ddots & \vdots \\ v_{p,1} & \cdots & v_{p,q} \end{pmatrix}$$

Entries are **variables**

$r \times r$ matrix: $A_{i,j}(t) = \sum_\ell \binom{p-i}{\ell}\binom{q-j}{\ell} t^\ell$

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Thom/Porteous 71, Giambelli 04, Harris/Tu 84

The **degree** of $\mathscr{D}$ is

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Conca/Herzog AMS'94, *Abhyankar '88*

The **Hilbert series** of $\mathscr{D}$ is

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Ingredients of the proof:

- **Cohen–Macaulay** rings;
- **quasi-homogeneous** polynomials.

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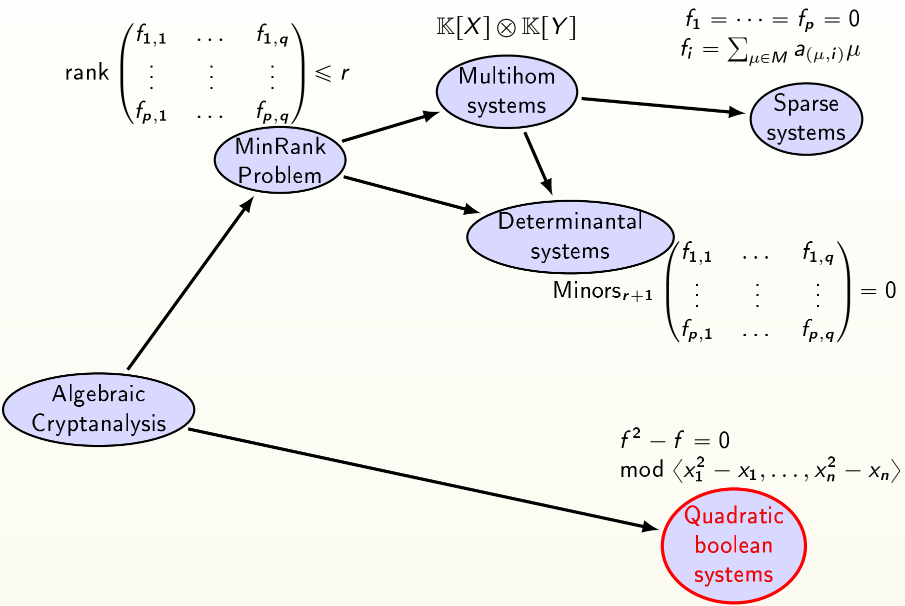

Quadratic boolean systems
with M. Bardet, J.-C. Faugère, B. Salvy

Boolean MQ Problem

$f_1, \ldots, f_m \in \mathbb{F}_2[x_1, \ldots, x_n]$ **quadratic polynomials.**
Find **one/all** **boolean solution** of the system

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- **NP-hard** problem $\rightsquigarrow$ SAT.
- Security of several modern **cryptosystems** relies on the difficulty of **Boolean MQ** (QUAD,...).
- Asymptotically, the number of solutions follows a **Poisson law** of parameter 2^{n-m} $\rightsquigarrow$ **few solutions** for **random** systems (*Fusco/Bach*, TAMC'07).
- Best proven **worst case complexity bound**: exhaustive search, $4 \cdot 2^n \log_2 n$ (*Bouillaguet/Chen/Cheng/Chou/Niederhagen/Shamir/Yang* CHES'10).

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- Best proven **worst case complexity bound**: exhaustive search, $4 \cdot 2^n \log_2 n$ (*Bouillaguet/Chen/Cheng/Chou/Niederhagen/Shamir/Yang* CHES'10).

Problem: construct an $O(2^{cn})$ algorithm, with $\mathbf{c < 1}$.

Related works

Algorithmic Tools

- Exhaustive search: *Bouillaguet/Chen/Cheng/Chou/Niederhagen/Shamir/Yang* CHES'10,...;
- SAT-Solvers: *Davis/Putnam/Logemann/Loveland* J. of ACM'60, Comm. of ACM'62;
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over $\mathbb{F}_2$, random systems $\neq$ generic systems

Main results

$$f_1, \ldots, f_m \in \mathbb{F}_2[x_1, \ldots, x_n].$$

Algorithm:
use (sparse) **linear algebra** to prune useless subtrees in the exhaustive search tree.

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Under precise *algebraic assumptions*, if $m = n$, the **complexity** is

- $O\left(2^{0.841n}\right)$ with a **deterministic** variant;
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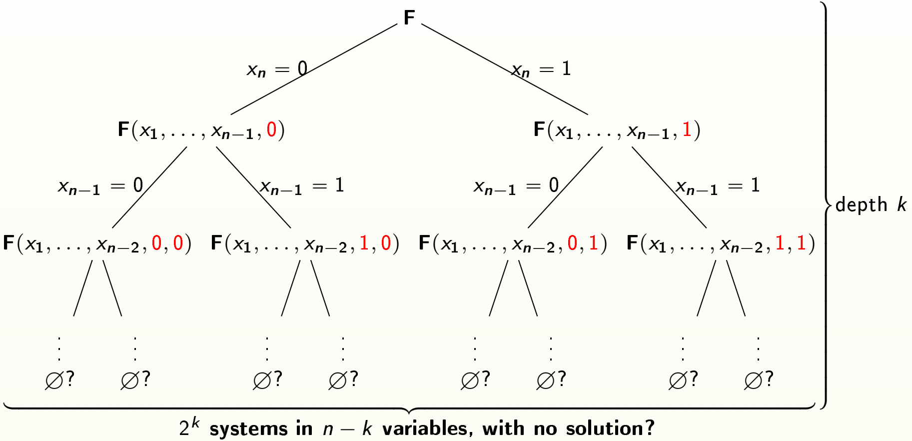
\+ **generalizations** when $m = \alpha n$ $(\alpha \geqslant 1)$.

Algebraic assumptions: variant of **Fröberg Conjecture** on the algebraic structure of generic overdetermined systems.

Algorithm



F
$x_n = 0$
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depth k
2^k **systems in** $n - k$ **variables, with no solution?**

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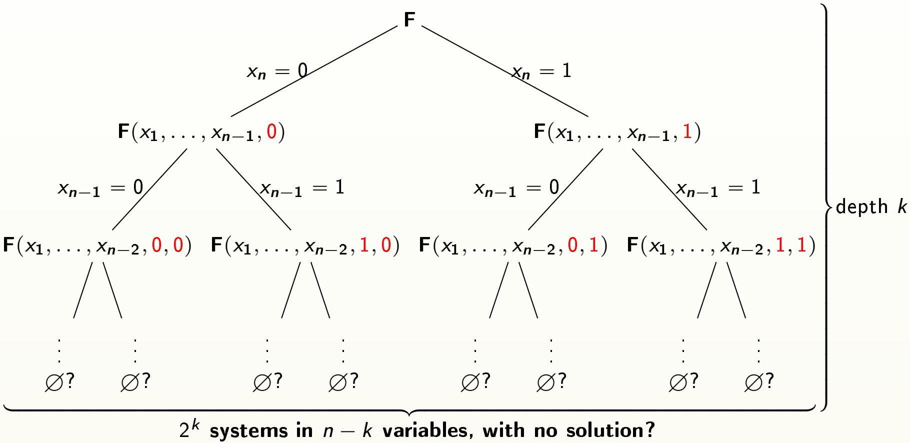

Hilbert Nullstellensatz

$\mathbf{F}(x_1, \ldots, x_{n-k}, a_{n-k+1}, \ldots, a_n)$ has no solution in $\mathbb{F}_2^{n-k}$

$$\Updownarrow$$

$$1 \in \langle \mathbf{F}, x_1^2 - x_1, \ldots, x_n^2 - x_n \rangle$$

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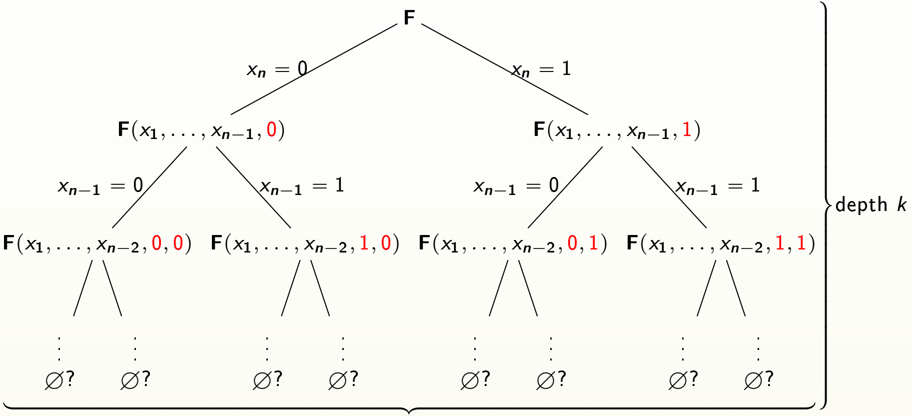

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Can be tested by solving **a linear system** involving the **Macaulay matrix**

Boolean Macaulay matrix in degree d

$$I = \langle f_1, \ldots, f_m \rangle \subset \mathbb{F}_2[x_1, \ldots, x_n].$$

Rows: all products tf_i where $t \in \text{SquareFreeMonomials}(d-2)$.
Columns: Square-free monomials of degree at most d.

$$\begin{array}{c} \\ t_1 f_1 \\ \vdots \\ t_k f_m \end{array} \begin{array}{c} m_1 > \cdots > m_\ell \\ \left(\begin{array}{ccc} & & \\ & & \\ & & \end{array} \right) \end{array} = \text{Mac}$$

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Problem: which d?

Algorithm BooleanSolve

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For all $(a_{n-k+1}, \dots, a_n) \in \mathbb{F}_2^k$
For i from 1 to m **(specialization)**
$\tilde{f}_i(x_1, \dots, x_{n-k}) := f_i(x_1, \dots, x_{n-k}, a_{n-k+1}, \dots, a_n) \in \mathbb{F}_2[x_1, \dots, x_{n-k}]$.
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    EndFor
    M := **boolean Macaulay matrix** of $(\tilde{f}_1, \ldots, \tilde{f}_m)$ in degree d_0.
    If the system $\mathbf{u} \cdot \mathrm{M} = \begin{pmatrix} 0 & \ldots & 0 & 1 \end{pmatrix}$ is inconsistent **(pruning)**
        $T :=$ solutions of the system $(\tilde{f}_1 = \cdots = \tilde{f}_m = 0)$ (exhaustive search).
        For all $(t_1, \ldots, t_{n-k}) \in T$
            $S := S \cup \{(t_1, \ldots, t_{n-k}, a_{n-k+1}, \ldots, a_n)\}$.
        EndFor
    EndIf
EndFor
Return S.

Workplan

1. Choice of d_0 (in function of the number of specialized variables k)?
   $\rightsquigarrow$ index of the **first non-positive coefficient** in $\frac{(1+t)^{n-k}}{(1-t)(1+t^2)^m}$
   $\rightsquigarrow$ $d_0 \sim M(\gamma)n$ when $k = (1-\gamma)n$

2. Sizes of the **Macaulay matrices** (function of k)?

3. Complexity of the **consistency tests** (function of k)?
   $O(2^{(1-\gamma+\omega F(\gamma)+\varepsilon)n})$

4. Find optimal k for **asymptotic complexity**?
   - Gauss: $k = 0.73n$;
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Experiments

- **Algebraic assumptions** are verified with prob. close to 1.
- Probabilistic variant: when $n = m$, **more efficient** than exhaustive search when $n \geqslant 200$ $\rightsquigarrow$ Crypto applications (QUAD).

Variant of Fröberg conjecture

The **proportion** of γ-strong semi-regular systems tends to 1 when $n \to \infty$.

Solving αn equations in n variables: $2^{c\,n}$

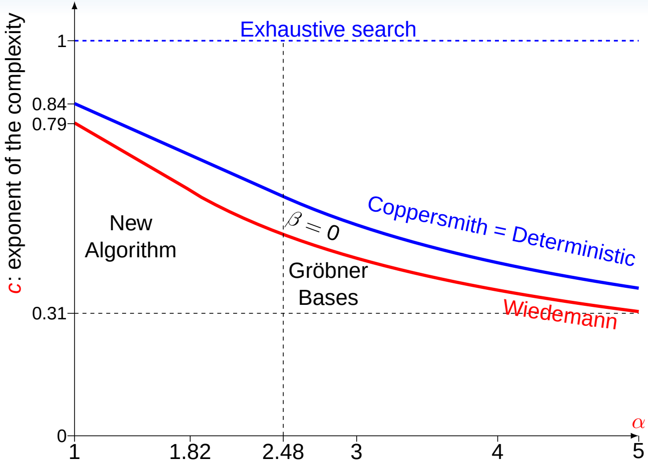

Figure: Exponent of the complexity in terms of α

Conclusion and Perspectives

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Algorithmic improvements

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  [Crypto 2008] **328233s** ⟶ [Issac 2010] **935s** ⟶ [Pasco 2010] **73s**
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Perspectives

- Dedicated algorithm for **determinantal systems**?

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- **Systematic methodologies** for the analysis of **structures** (Hilbert series, degree of regularity, tools for commutative algebra and algebraic geometry, invariants,. . .).

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Thank you!