Computing the ρ constant

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Abstract

At the 4th Number Theory Down Under conference (Newcastle, Australia) in September 2016, Timothy Trudgian gave a talk entitled "A Tale of Two Omegas" during which he challenged the audience to give the best possible approximation of a constant ρ. The winner of the bet was Shi Bai who proposed $\rho \approx 0.75$ and won 9 Australian dollars. We give a closed formula for ρ, and a 1000-digit approximation.

For any positive integer n, let $\omega(n)$ be the number of distinct prime factors of n, and $\Omega(n)$ be the number of prime factors of n counting multiplicities. We want to study the density ρ of integers n such that $\omega(n) \equiv \Omega(n) \pmod 2$.

Writing p_i the i-th prime number, we denote by δ_i the density of integers exactly divisible by a positive even power of p_i. We have

$$\delta_i = \frac{1}{p_i^2} - \frac{1}{p_i^3} + \frac{1}{p_i^4} - \cdots = \sum_{k \geq 2} \left(-\frac{1}{p_i}\right)^k = \frac{1}{p_i^2} \sum_{k \geq 0} \left(-\frac{1}{p_i}\right)^k = \frac{1}{p_i(p_i+1)}.$$

We can extend the definitions of ω and Ω as $\omega_i(n)$ the number of distinct prime factors of n up to and including p_i, and $\Omega_i(n)$ the number of prime factors of n counting multiplicities up to and including p_i.

Let us now define ρ_i as the density of integers n such that $\omega_i(n) \equiv \Omega_i(n) \pmod 2$. Clearly, as $\omega_0(n) = \Omega_0(n) = 0$ for all n, $\rho_0 = 1$. Furthermore, given ρ_i, we can compute ρ_{i+1} as $\rho_{i+1} = \rho_i(1 - \delta_{i+1}) + (1 - \rho_i)\delta_{i+1}$ since, for any integer n such that $\omega_{i+1}(n) \equiv \Omega_{i+1}(n) \pmod 2$, we have:

- either $\omega_i(n) \equiv \Omega_i(n) \pmod 2$ and n is not exactly divisible by a positive even power of p_{i+1} (*i.e.*, the exponent of p_{i+1} in the factorization of n is either zero or an odd integer);
- or $\omega_i(n) \not\equiv \Omega_i(n) \pmod 2$ and n is exactly divisible by a positive even power of p_{i+1} (*i.e.*, the exponent of p_{i+1} in the factorization of n is a non-zero even integer).

For instance, we have $\rho_1 = 5/6$, $\rho_2 = 7/9$, $\rho_3 = 41/54$, etc.

Expanding and rewriting the recurrence relation, we obtain that

$$\rho_i = \frac{1}{2}\left(1 + \prod_{1 \leq j \leq i} (1 - 2\delta_j)\right) = \frac{1}{2}\left(1 + \prod_{1 \leq j \leq i} \left(1 - \frac{2}{p_j(p_j+1)}\right)\right).$$

Proof. Trivial for ρ_0. By induction, assuming that $\rho_i = \frac{1}{2}\left(1 + \prod_{1 \leq j \leq i} (1 - 2\delta_j)\right)$, we have

$$\begin{aligned}\rho_{i+1} &= \rho_i(1 - \delta_{i+1}) + (1 - \rho_i)\delta_{i+1} = \rho_i(1 - 2\delta_{i+1}) + \delta_{i+1} \\ &= \tfrac{1}{2}\left(1 - 2\delta_{i+1} + \textstyle\prod_{1 \leq j \leq i+1} (1 - 2\delta_j)\right) + \delta_{i+1} = \tfrac{1}{2}\left(1 + \textstyle\prod_{1 \leq j \leq i+1} (1 - 2\delta_j)\right).\end{aligned}$$

□

We want to compute ρ, the limit of ρ_i when i tends to infinity. This requires evaluating the Euler-type product $\prod_p(1 - 2/(p(p+1)))$. This product is related to the *strongly carefree* constant $K_2 = \zeta(2)^{-2}\prod_p(1 - 1/(p+1)^2)$ [4], since we have, for all p,

$$\left(1 - \frac{1}{(p+1)^2}\right)\left(1 - \frac{1}{p^2}\right) = 1 - \frac{2}{p(p+1)},$$

whence

$$\prod_p \left(1 - \frac{2}{p(p+1)}\right) = \zeta(2)^2 K_2 \cdot \prod_p \left(1 - \frac{1}{p^2}\right) = \zeta(2) K_2 \qquad \text{and} \qquad \rho = \frac{1}{2}(1 + \zeta(2)K_2).$$

From $\zeta(2) = \pi^2/6$ and the approximation of K_2 given in [5], we get

$$\rho \approx 0.73584030680649893403761781654 0\ldots.$$

Using the method described in [3], we wrote a C program using the GMP [2] and MPFR [1] libraries to obtain 1000 decimal digits of ρ in 2 seconds on a modern computer:

$$\begin{aligned}\rho \approx 0.&7358403068064989340376178165402410437129631910034934418178686277088660 58\backslash\\ &37984137204810501368847451151565056612284119010240461121168788404604 4892\backslash\\ &90436208640046435701050404964067641661147608413571726388653556364380 5760\backslash\\ &64422194189272350859691721446456014521830342346510552336534389439987 6950\backslash\\ &29090851772584341283680479687355747450341578412027310083185240181507 1884\backslash\\ &75247465199815551061430494372333056419870881805875185962569215870646 4491\backslash\\ &11892561493389497666455467533272416086453799520366149142679007387632 5701\backslash\\ &42209664283308834514516635682293369852419440515760818289024874633174 7324\backslash\\ &54686439967918935105319085857655957533138621714848761163283383730300 3554\backslash\\ &77240060622882960681586493325411317873541043127459820599296122014981 6699\backslash\\ &95757339223147843752950368518960994498330617685749024259836276957872 0704\backslash\\ &71543422085782618429623728209316602388811343704306623184933279119612 5464\backslash\\ &59366235297883474645077309181385296279518004017108481635970179847444 1936\backslash\\ &8394121574471862067394065063161173529462108558444754318971609164\ldots.\end{aligned}$$

We give below a few benchmarks for various precisions, with parameters n and M as described in [3]: n is the number of terms of the Euler product which are actually computed, and M is the cut-off value used when approximating the remaining terms $\prod_{p>p_n}(1 - 2/(p(p+1)))$ as $\prod_{i=2}^{M}\zeta_n(i)^{e(i)}$, where $\zeta_n(i)$ is as in [3].

Decimal digits	n	M	**Working precision**	**Approximation error**	**Rounding error**	**CPU time**
1 000	200	360	3 694 bits	$1.09 \cdot 10^{-1003}$	$2.38 \cdot 10^{-1004}$	2 s
2 000	200	719	7 378 bits	$9.05 \cdot 10^{-2005}$	$1.41 \cdot 10^{-2005}$	17 s
5 000	400	1595	18 220 bits	$3.65 \cdot 10^{-5005}$	$1.09 \cdot 10^{-5005}$	362 s
10 000	800	2869	36 102 bits	$4.95 \cdot 10^{-10005}$	$3.78 \cdot 10^{-10005}$	4007 s

References

[1] Laurent Fousse, Guillaume Hanrot, Vincent Lefèvre, Patrick Pélissier, and Paul Zimmermann. MPFR: A multiple-precision binary floating-point library with correct rounding. *ACM Transactions on Mathematical Software*, 33(2):article 13, 2007.

[2] Torbjörn Granlund and the GMP development team. *GNU MP: The GNU Multiple Precision Arithmetic Library*, 6.1.1 edition, 2016. `http://gmplib.org/`.

[3] Pieter Moree. Approximation of singular series and automata. *Manuscripta Math.*, 101:385–399, 2000.

[4] Pieter Moree. Counting carefree couples. Available at `http://arxiv.org/abs/math.NT/0510003`, 2005.

[5] Gerhard Niklasch. Some number-theoretical constants. Available at `https://oeis.org/A001692/a001692.html`, 2002.